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MÖBIUS TRANSFORMATIONS OF THE DISC AND ONE-PARAMETER GROUPS OF ISOMETRIES OF H^p

BY

EARL BERKSON, ROBERT KAUFMAN AND HORACIO PORTA[1]

ABSTRACT. Let $\{T_t\}$ be a strongly continuous one-parameter group of isometries in H^p $(1 \leq p < \infty, p \neq 2)$ with unbounded generator. There is a uniquely determined one-parameter group $\{\phi_t\}$, $t \in \mathbf{R}$, of Möbius transformations of the (open) disc D corresponding to $\{T_t\}$ by way of Forelli's theorem. The interplay between $\{T_t\}$ and $\{\phi_t\}$ is studied, and the spectral properties of the generator A of $\{T_t\}$ are analyzed in this context. The nature of the set S of common fixed points of the functions ϕ_t plays a crucial role in determining the behavior of A. The spectrum of A, which is a subset of $i\mathbf{R}$, can be a discrete set, a translate of $i\mathbf{R}_+$ or of $i\mathbf{R}_-$, or all of $i\mathbf{R}$. If S is not a doubleton subset of the unit circle, $\{T_t\}$ can be extended to a holomorphic semigroup of H^p-operators, the semigroup being defined on a half-plane. The treatment of $\{T_t\}$ is facilitated by developing appropriate properties of one-parameter groups of Möbius transformations of D. In particular, such groups are in one-to-one correspondence (via an initial-value problem) with the nonzero polynomials q, of degree at most 2, such that $\operatorname{Re}[\bar{z}q(z)] = 0$ for all unimodular z. A has an explicit description (in terms of the polynomial corresponding to $\{\phi_t\}$) as a differential operator.

0. Introduction. The purpose of this paper is to investigate strongly continuous one-parameter groups of isometries (and their generators) in the spaces H^p (of the disc), $1 \leq p < \infty$. Such groups were studied in [1], where basic facts were developed. The following is of central importance to our subject [4, Theorem 2].

(0.1) PROPOSITION. *If T is a linear isometry of H^p onto H^p, $1 \leq p < \infty$, $p \neq 2$, then there are a unimodular constant ν and a Möbius transformation of the disc ϕ such that*

$$(0.2) \qquad (Tf)(z) = \nu[\phi'(z)]^{1/p} f(\phi(z)), \quad f \in H^p, \ |z| < 1.$$

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On the other hand, if $1 \leq p < \infty$, (0.2) *defines a linear isometry of* H^p *onto* H^p.

In §2 we take up the relationship between groups of H^p-isometries and one-parameter groups of Möbius transformations of D $(= \{z \in \mathbf{C}: |z| < 1\})$. A one-parameter group of Möbius transformations of the disc is a homomorphism $t \longmapsto \phi_t$ of the additive group of $\mathbf{R}$ into the group M (under composition) of all univalent analytic mappings of D onto D such that, for each $z_0 \in D$, $\phi_t(z_0)$ is a continuous function of t and, for some $u \in \mathbf{R}$, ϕ_u is not the identity map.

One-parameter groups of Möbius transformations of D are closely associated with polynomials of a certain type, and this association is developed in §1 (see Theorem (1.5)). §1 also deals with the classification of such groups, as well as their extensions to groups of linear fractional transformations (see Theorem (1.10)).

In §§3 and 4 we consider the generators of one-parameter groups of H^p-isometries, and present a fairly comprehensive treatment of the spectral properties of such a generator.

Throughout what follows, we shall denote the unit circle $\{z \in \mathbf{C}: |z| = 1\}$ by C, the extended complex plane by $\mathbf{C}_e$, and the composition of two mappings f and g by $f \circ g$. We usually deal with $H^p(D)$. When, on occasion, we pass to the boundary C, this fact will be made clear by the context.

We remark that for $1 \leq p < \infty$, $p \neq 2$, it is known [1, Theorem (2.8)] that the one-parameter groups of H^p-isometries continuous in the uniform operator topology are precisely the trivial groups of the form $\{e^{i\beta t}I\}$, $t \in \mathbf{R}$, where I is the identity operator, and β is a real constant.

1. One-parameter groups of Möbius transformations of the disc and their extensions. Let $\{\phi_t\}$, $t \in \mathbf{R}$, be a one-parameter group of Möbius transformations of D. It will sometimes be convenient to write $\phi(t, z)$ in place of $\phi_t(z)$ (for $t \in \mathbf{R}$, $z \in \mathbf{C}$). In this notation, the partial derivative at (r, z_0) of $\phi(\cdot, \cdot)$ with respect to t (resp., with respect to z) will be denoted by $\phi_1(r, z_0)$ (resp., $\phi_2(r, z_0)$), provided it exists. As observed in [1, §1], $\phi_1(r, z_0)$ and $\phi_2(r, z_0)$ exist provided z_0 is not a pole of the Möbius transformation $\phi_r(\cdot)$. The following fact will be basic to our considerations [1, Proposition (1.5)]:

(1.1) SCHOLIUM. *The set* S *of common fixed points in* $\mathbf{C}_e$ *of the functions* ϕ_t, $t \in \mathbf{R}$, *must be one of the following*: (i) *a doubleton set consisting of a point* τ *in* D *and of* $\bar{\tau}^{-1}$ *(the latter taken to be* ∞ *if* $\tau = 0$), (ii) *a singleton subset of* C, *or* (iii) *a doubleton subset of* C. *If* $u \in \mathbf{R}$ *and* $\phi_u(\cdot)$ *is not the identity map, then* S *coincides with the set of all fixed points in* $\mathbf{C}_e$ *of* $\phi_u(\cdot)$.

As in [1], $\{\phi_t\}$ will be said to be of type (i), (ii), or (iii) according as (i), (ii), or (iii) of the mutually exclusive descriptions for S in (1.1) holds. A further basic fact from [1, (1.6) and (1.7)] is that $\phi_1(0, z)$ is a nonzero polynomial in z of degree one or two whose set of zeros is $S \cap \mathbf{C}$. We denote $\phi_1(0, \cdot)$ by $q(\cdot)$ and call it the invariance polynomial of $\{\phi_t\}$, $t \in \mathbf{R}$.

We shall now characterize one-parameter groups of Möbius transformations of D by a suitable type of initial-value problem. For such a group $\{\phi_t\}$, $|\phi(t, z)|^2 = 1$ for $t \in \mathbf{R}$, $z \in C$. If we partially differentiate this identity with respect to t and then set $t = 0$, we get

$$\text{(1.2)} \qquad \operatorname{Re}[\bar{z}q(z)] = 0 \quad \text{for all } z \text{ in } C.$$

Let $\mathcal{I}$ be the set of all nonzero polynomials f of degree at most 2 such that $\operatorname{Re}[\bar{z}f(z)] = 0$ for all unimodular z. A simple calculation based on the definition and the linear independence on C of the functions z, 1, and $\bar{z}$ shows that $\mathcal{I}$ consists of all polynomials f of the form $f(z) = Az^2 + iBz - \bar{A}$, where A is a complex constant, B is a real constant, and $|A|^2 + B^2 > 0$.

Next we observe that for the group $\{\phi_t\}$ we have for all $s, t \in \mathbf{R}$ and $z \in \bar{D}$, $\phi(s + t, z) = \phi(s, \phi(t, z))$. Partial differentiation with respect to s, followed by setting $s = 0$, gives $\phi_1(t, z) = q(\phi(t, z))$ for $t \in \mathbf{R}$, $z \in \bar{D}$.

Conversely, let $p_0 \in \mathcal{I}$, let J be an interval of $\mathbf{R}$ containing 0, and consider the initial-value problem on $J \times \bar{D}$

$$\text{(1.3)} \qquad \partial\psi(t, z)/\partial t = p_0(\psi(t, z)), \qquad \psi(0, z) = z.$$

Regarding (1.3) as an ordinary differential equation for each fixed $z \in \bar{D}$, we get from a standard uniqueness theorem that (1.3) has at most one solution for ψ on $J \times \bar{D}$, and the Picard method of successive approximations shows that there is a $\delta > 0$ such that (1.3) has a solution on $[-\delta, \delta] \times \bar{D}$ which is the uniform limit on $[-\delta, \delta] \times \bar{D}$ of a sequence of polynomials in t and z. Moreover, if $\alpha > 0$, and ψ is a solution of (1.3) on $[-\alpha, \alpha] \times \bar{D}$, then for fixed $z_0 \in \bar{D}$

$$\text{(1.4)} \qquad \frac{d}{dt}[|\psi(t, z_0)|^2 - 1] = 2 \operatorname{Re}[\overline{\psi(t, z_0)}p_0(\psi(t, z_0))] \quad \text{on } [-\alpha, \alpha].$$

For $w \in \mathbf{C}$ let w^* be $w/|w|$ if $w \neq 0$, and 1 if $w = 0$. Then

$$p_0(\psi(t, z_0)) - p_0(\psi(t, z_0)^*) = O(|\psi(t, z_0) - \psi(t, z_0)^*|) \quad \text{on } [-\alpha, \alpha].$$

Thus (1.4) and the definition of $\mathcal{I}$ give us that

$$\left|\frac{d}{dt}[|\psi(t, z_0)|^2 - 1]\right| = O(||\psi(t, z_0)|^2 - 1|).$$

A standard Gronwall type argument allows us to conclude from this fact that, for $t \in [-\alpha, \alpha]$, $|\psi(t, z)| \leqslant 1$ (resp., $|\psi(t, z)| = 1$) for $z \in \bar{D}$ (resp., for $|z| = 1$).

It is easy to see with the aid of this boundedness conclusion that (1.3) has a unique solution ϕ on $\mathbf{R} \times \bar{D}$. As usual, for $t \in \mathbf{R}$ we write ϕ_t for the function $\phi(t, \cdot)$ on $\bar{D}$. In particular each ϕ_t maps $\bar{D}$ into $\bar{D}$ and C into C. Since the equation in (1.3) is autonomous, we get that $\phi_{t+s}(z) = \phi_t(\phi_s(z))$ for $t, s \in \mathbf{R}$, $z \in \bar{D}$. Our earlier observation concerning the Picard method assures us of an interval $[-\delta, \delta]$ such that, for each t in $[-\delta, \delta]$, ϕ_t is continuous on $\bar{D}$ and analytic on D. It is now clear that, for $t \in [-\delta, \delta]$, ϕ_t is a one-to-one analytic map of D onto D and must be a Möbius transformation of the disc. Since $\phi_{u+v} = \phi_u \circ \phi_v$ for all $u, v \in \mathbf{R}$, it follows easily that ϕ_u is a Möbius transformation of the disc for each $u \in \mathbf{R}$. Moreover, it is immediate from (1.3) that p_0 is the invariance polynomial of the one-parameter group $\{\phi_t\}$ of Möbius transformations of the disc. We have established:

(1.5) THEOREM. *The correspondence which assigns to each one-parameter group of Möbius transformations of the disc its invariance polynomial is a one-to-one map of the set of all such groups onto* $\mathcal{I}$. *The inverse of this map assigns to each* $p_0 \in \mathcal{I}$ *the unique solution on* $\mathbf{R} \times \bar{D}$ *of the initial-value problem*

$$\partial\psi(t, z)/\partial t = p_0(\psi(t, z)), \qquad \psi(0, z) = z.$$

The next theorem makes explicit the correspondence of Theorem (1.5). Some additional notation will be helpful. Let $\mathbf{R}_0$ (resp., P) denote the set of nonzero (resp., positive) real numbers. Let S_1 (resp., S_2) be the Cartesian product $\mathbf{R}_0 \times D$ (resp., $\mathbf{R}_0 \times C$). Let S_3 be the set of all ordered triples (c, α, β) such that $c \in P$, $\alpha \in C, \beta \in C$, $\alpha \neq \beta$. Let G_1 (resp., G_2, G_3) be the set of all one-parameter groups of Möbius transformations of the disc of type (i) (resp., type (ii), type (iii)). For $\tau \in D$, let γ_τ be the Möbius transformation of the disc given by $\gamma_\tau(z) = (z - \tau)/(\bar{\tau}z - 1)$. (Note that γ_τ is its own inverse map.) For $(\alpha, \beta) \in C \times C$ with $\alpha \neq \beta$, let $\sigma_{\alpha,\beta}$ be the linear fractional transformation given by $\sigma_{\alpha,\beta}(z) = (z - \alpha)/(z - \beta)$.

(1.6) THEOREM. (i) *To each ordered pair* (c, τ) *in* S_1 *there corresponds a group* $\{\phi_t\}$ *belonging to* G_1 *given by*

$$\phi_t(z) = \gamma_\tau(e^{ict}\gamma_\tau(z)) \quad \textit{for} \quad t \in \mathbf{R}, \ z \in D. \tag{1.7}$$

This correspondence is a one-to-one map of S_1 *onto* G_1. *If* $\{\phi_t\} \in G_1$ *has the unique representation* (1.7), *then its invariance polynomial is* icz (*resp.,* $ic\bar{\tau}(|\tau|^2 - 1)^{-1}(z - \tau)(z - 1/\bar{\tau})$) *if* $\tau = 0$ (*resp.,* $\tau \neq 0$).

(ii) *To each ordered pair* (c, α) *in* S_2 *there corresponds a group* $\{\phi_t\}$ *belonging to* G_2 *given by*

(1.8) $$\phi_t(z) = \frac{(1 - ict)z + ict\alpha}{-ic\bar{\alpha}tz + (1 + ict)} \quad \textit{for} \quad t \in \mathbf{R}, \ z \in D.$$

This correspondence is a one-to-one map of S_2 *onto* G_2. *If* $\{\phi_t\} \in G_2$ *has the unique representation* (1.8), *then its invariance polynomial is* $ic\bar{\alpha}(z - \alpha)^2$.

(iii) *To each ordered triple* (c, α, β) *in* S_3 *there corresponds a group* $\{\phi_t\}$ *belonging to* G_3 *given by*

(1.9) $$\phi_t(z) = (\sigma_{\alpha,\beta})^{-1}(e^{ct}\sigma_{\alpha,\beta}(z)) \quad \textit{for} \quad t \in \mathbf{R}, \ z \in D.$$

This correspondence is a one-to-one map of S_3 *onto* G_3. *If* $\{\phi_t\} \in G_3$ *has the unique representation* (1.9), *then its invariance polynomial is* $(c/(\alpha - \beta))(z - \alpha)(z - \beta)$.

PROOF. We begin with (iii). Let $(c, \alpha, \beta) \in S_3$. Note that $\sigma_{\alpha,\beta}(\bar{D})$ is the half-plane H (including the point at ∞) given by $\mathrm{Re}[(1 - \bar{\alpha}\beta)z] \leq 0$. Thus if we define ϕ_t by (1.9), then ϕ_t is a linear fractional transformation with $\phi_t(\bar{D}) = (\sigma_{\alpha,\beta})^{-1}(H) = \bar{D}$. Clearly $\{\phi_t\}$, $t \in \mathbf{R}$, is a one-parameter group of Möbius transformations of the disc having α and β as common fixed points. Conversely, let $\{\phi_t\}$ be of type (iii) with distinct common fixed points $\alpha, \beta \in C$. It follows from elementary facts about linear fractional transformations [7, p. 323] that for each $t \in \mathbf{R}$ there is a unique nonzero complex constant K_t such that $\sigma_{\alpha,\beta}(\phi_t(z)) = K_t\sigma_{\alpha,\beta}(z)$. Moreover, $K_{t+s} = K_tK_s$ for all $t, s \in \mathbf{R}$. Since $\phi_t(z)$ is continuous in t for each $z \in D$, K_t is a continuous function of t. If H is the half-plane mentioned above, then for each $t \in \mathbf{R}$, $H = K_tH$, which implies $K_t > 0$. It is now clear that there is a real number λ such that $K_t = e^{\lambda t}$ for all $t \in \mathbf{R}$. Since $\{\phi_t\}$ is not a constant function of t, $\lambda \neq 0$. By taking reciprocals, if necessary, on both sides of the equation

$$\frac{\phi_t(z) - \alpha}{\phi_t(z) - \beta} = e^{\lambda t}\frac{z - \alpha}{z - \beta}, \qquad t \in \mathbf{R}, \ z \in D,$$

we conclude that $\{\phi_t\}$ corresponds to an ordered triple in S_3. If (c_1, α_1, β_1) and (c_2, α_2, β_2) belong to S_3 and define the same group $\{\phi_t\}$ by (1.9), then it is obvious from (1.9) that the set of common fixed points of $\{\phi_t\}$, $t \in \mathbf{R}$, coincides with each of the sets $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$. If $\alpha_1 = \beta_2$ and $\beta_1 = \alpha_2$, then it follows readily that $e^{-c_2t} = e^{c_1t}$ for all $t \in \mathbf{R}$, which gives the conclusion $-c_2 = c_1$. But $c_1 > 0$ and $c_2 > 0$. Thus $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, whence $c_1 = c_2$. To compute the invariance polynomial of $\{\phi_t\}$ in (1.9), we first differentiate with respect to t on both sides of the equation $\sigma_{\alpha,\beta}(\phi_t(z)) = e^{ct}\sigma_{\alpha,\beta}(z)$ to get

$$\sigma'_{\alpha,\beta}(\phi(t, z))\, \partial\phi(t, z)/\partial t = ce^{ct}\sigma_{\alpha,\beta}(z).$$

Dividing the latter equation by the former gives

$$\frac{\sigma'_{\alpha,\beta}(\phi(t, z))}{\sigma_{\alpha,\beta}(\phi(t, z))} \frac{\partial\phi(t, z)}{\partial t} = c.$$

By direct calculation,

$$\frac{\sigma_{\alpha,\beta}(z)}{\sigma'_{\alpha,\beta}(z)} = \frac{1}{\alpha - \beta}(z - \alpha)(z - \beta) \quad \text{for } z \in D.$$

Thus

$$\frac{\partial\phi(t, z)}{\partial t} = \frac{c}{\alpha - \beta}(\phi(t, z) - \alpha)(\phi(t, z) - \beta) \quad \text{for } z \in D,$$

and the desired conclusion follows. The proof of (i) (except for the trivial parts when $\tau = 0$) is entirely analogous to that of (iii) (using τ and $\bar{\tau}^{-1}$ in place of α and β). Also, (i) has been tabulated in [1, Theorem (1.10) and Corollary (1.13)], and we omit the details of its proof. It remains to establish (ii). Obviously the invariance polynomials of the groups of type (ii) are precisely the polynomials belonging to I with a double root belonging to C. The latter are, in turn, precisely the polynomials of the form $q(z) = ic\bar{\alpha}(z - \alpha)^2$ for c real and nonzero and $\alpha \in C$. The proof of (ii) is now easily completed by explicitly solving, for such a polynomial q, the initial-value problem of Theorem (1.5), thereby obtaining the corresponding group, $\{\phi_t\}$, $t \in \mathbf{R}$. The solution guaranteed by Theorem (1.5) can be found on $\mathbf{R} \times D$ by separation of variables.

In the next section we shall be concerned with extending, whenever possible, a one-parameter group of isometries of H^p to a semigroup (defined on a half-plane) of operators on H^p. In preparation for this we now take up the related problem of extending a one-parameter group of Möbius transformations of the disc to a planar group of linear fractional transformations in the following sense.

DEFINITION. A planar group $\{L_w\}$, $w \in \mathbf{C}$, of linear fractional transformations is a homomorphism $w \mapsto L_w$ of the additive group of $\mathbf{C}$ into the group (under composition) of all linear fractional transformations such that, for each $z \in \mathbf{C}$, $w \longrightarrow L_w(z)$ is a continuous mapping of $\mathbf{C}$ into $\mathbf{C}_e$, for some w L_w is not the identity map, and, for each $z \in \mathbf{C}$, $dL_w(z)/dw|_{w=0}$ exists. The planar group $\{L_w\}$, $w \in \mathbf{C}$, is said to extend a one-parameter group $\{\phi_t\}$, $t \in \mathbf{R}$, of Möbius transformations of the disc provided $L_t = \phi_t$ for $t \in \mathbf{R}$.

(1.10) THEOREM. *Every one-parameter group* $\{\phi_t\}$ *of Möbius transformations of the disc can be uniquely extended to a planar group* $\{\Phi_w\}$, $w \in \mathbf{C}$, *of linear fractional transformations. In fact*:

(i) *If* $\{\phi_t\}$ *is of type* (i) *with the representation* (1.7), *then* $\{\Phi_w\}$, $w \in \mathbf{C}$, *has the form*

$$\Phi_w(z) = \gamma_\tau(e^{icw}\gamma_\tau(z)) = \frac{(e^{icw} - |\tau|^2)z + \tau(1 - e^{icw})}{\tau(e^{icw} - 1)z + 1 - |\tau|^2 e^{icw}}. \tag{1.11}$$

(ii) *If* $\{\phi_t\}$ *is of type* (ii) *with the representation* (1.8), *then* $\{\Phi_w\}$, $w \in \mathbf{C}$, *has the form*

$$\Phi_w(z) = \frac{(1 - icw)z + icw\alpha}{-ic\bar{\alpha}wz + 1 + icw}. \tag{1.12}$$

(iii) *If* $\{\phi_t\}$ *is of type* (iii) *with the representation* (1.9), *then* $\{\Phi_w\}$, $w \in \mathbf{C}$, *has the form*

$$\Phi_w(z) = (\sigma_{\alpha,\beta})^{-1}(e^{cw}\sigma_{\alpha,\beta}(z)) = \frac{(\beta e^{cw} - \alpha)z + \alpha\beta(1 - e^{cw})}{(e^{cw} - 1)z + \beta - \alpha e^{cw}}. \tag{1.13}$$

PROOF. One verifies readily that the middle and right-hand members in each of (1.11) and (1.13) are equal, and that (1.12) defines a planar group $\{\Phi_w\}$, $w \in \mathbf{C}$, which obviously extends the group $\{\phi_t\}$ in (ii). It is obvious that (1.11) (resp., (1.13)) defines a planar group extending the group $\{\phi_t\}$ in (i) (resp., (iii)). To prove uniqueness, let $\{\phi_t\}$ be a one-parameter group of Möbius transformations of D with invariance polynomial q, and let $\{\Psi_w\}$ be a planar group of linear fractional transformations extending $\{\phi_t\}$. It follows from definitions that if $(w_0, z_0) \in \mathbf{C} \times \mathbf{C}$ and z_0 is not a pole of Ψ_{w_0}, then $d\Psi_w(z_0)/dw|_{w=w_0}$ exists and equals $q(\Psi_{w_0}(z_0))$. Thus if $v_0 \in \mathbf{R}$ and z_0 is not a pole of Ψ_{iv_0}, then $d\Psi_{iv}(z_0)/dv|_{v=v_0} = iq(\Psi_{iv_0}(z_0))$. The Picard method and a standard uniqueness theorem show that there is a $\delta > 0$ such that for each $z \in \bar{D}$, the initial value problem $dy/dv = iq(y)$, $y(0) = z$, has a unique solution on every subinterval containing 0 of $[-\delta, \delta]$ (δ is independent of $z \in \bar{D}$). The desired uniqueness of extension now follows readily.

REMARKS. (1) The uniqueness of the representation for $\{\Phi_w\}$ in (1.11) (resp., (1.12), (1.13)) follows from the uniqueness of the representation for $\{\phi_t\}$ in (1.7) (resp., (1.8), (1.9)).

(2) If $\{L_w\}$ is a planar group of linear fractional transformations, then an argument similar to the one used in proving [1, Proposition (1.5)] shows that for any complex number w_0 such that L_{w_0} is not the identity, the set of fixed points in $\mathbf{C}_e$ of L_{w_0} coincides with the set of common fixed points in $\mathbf{C}_e$ of the functions L_w, $w \in \mathbf{C}$. In particular, if $\{L_w\}$ extends a one-parameter group $\{\phi_t\}$ of Möbius transformations of D, then, by the last sentence in (1.1), the set of common fixed points in $\mathbf{C}_e$ of the functions L_w for $w \in \mathbf{C}$ coincides with the set of common fixed points in $\mathbf{C}_e$ of the functions ϕ_t for $t \in \mathbf{R}$.

(1.14) THEOREM. *The linear fractional transformation* Φ_w *in* (1.11) (*resp.*, (1.12), (1.13)) *maps* D *into* D *if and only if* $c \operatorname{Im} w \geqslant 0$ (*resp.*, $c \operatorname{Im} w \leqslant 0$, $c \operatorname{Im} w = 2k\pi$ *for some integer* k).

PROOF. By the maximum principle $\Phi_w(D) \subseteq D$ if and only if $\Phi_w(\bar{D}) \subseteq \bar{D}$. Consider Φ_w in (1.11). Since γ_τ is a Möbius transformation of the disc, $\Phi_w(D) \subseteq D$ if and only if $e^{icw}D \subseteq D$. The assertion for this case is established. Consider next the group $\{\Phi_w\}$, $w \in \mathbf{C}$, in (1.12). For convenience, put the subscript zero on the constant c in (1.12) so that the real constant is now denoted by the symbol c_0. For arbitrary τ in D, $w \in \mathbf{C}$, let $\Phi_w^{(\tau)}$ be the linear fractional transformation in (1.11) with $c = c_0(|\tau|^2 - 1)$. Let $(\delta_{mn}(w))$ be the 2×2 matrix corresponding to the representation for Φ_w in (1.12), and let $(\delta_{mn}^{(\tau)}(w))$ be the matrix corresponding to the representation for $\Phi_w^{(\tau)}$ on the far right of (1.11). It is easy to see from the power series expansion for the exponential function that, uniformly on bounded subsets of the w-plane, we have, as $\tau \longrightarrow \alpha$,

$$\delta_{mn}^{(\tau)}(w) = (1 - |\tau|^2)\delta_{mn}(w) + o(1 - |\tau|^2) \quad \text{for } m, n = 1, 2. \tag{1.15}$$

If $c_0 \operatorname{Im} w \leqslant 0$ and $z \in \bar{D}$, then the already established first assertion of the theorem insures that $|\Phi_w^{(\tau)}(z)| \leqslant 1$. This fact together with (1.15) enables us to conclude that $\Phi_w(\bar{D}) \subseteq \bar{D}$ for $c_0 \operatorname{Im} w \leqslant 0$. On the other hand, if w is fixed with $c_0 \operatorname{Im} w > 0$, then from (1.11),

$$\gamma_\tau(\Phi_w^{(\tau)}(C)) = [\exp(c_0(1 - |\tau|^2) \operatorname{Im} w)]C.$$

If $z_0 \in C$, and z_0 is not a pole of Φ_w, then it follows from (1.15) that for $|\tau - \alpha|$ sufficiently small and positive, z_0 is not a pole of $\Phi_w^{(\tau)}$. Hence

$$|\Phi_w^{(\tau)}(z_0) - \tau| = |\bar{\tau}\Phi_w^{(\tau)}(z_0) - 1| \exp(c_0(1 - |\tau|^2) \operatorname{Im} w),$$

and so, as $\tau \longrightarrow \alpha$,

$$\begin{aligned} &|\Phi_w^{(\tau)}(z_0) - \tau|^2 - |\bar{\tau}\Phi_w^{(\tau)}(z_0) - 1|^2 \\ &\qquad = 2|\bar{\tau}\Phi_w^{(\tau)}(z_0) - 1|^2 c_0(1 - |\tau|^2) \operatorname{Im} w + o(1 - |\tau|^2). \end{aligned} \tag{1.16}$$

If we simplify (1.16), divide by $(1 - |\tau|^2)$, and let $\tau \longrightarrow \alpha$, we get

$$|\Phi_w(z_0)|^2 - 1 = 2|\bar{\alpha}\Phi_w(z_0) - 1|^2 c_0 \operatorname{Im} w. \tag{1.17}$$

Since α is a fixed point of Φ_w, if $z_0 \neq \alpha$ then the right-hand side of (1.17) is positive. Thus Φ_w does not map $\bar{D}$ into $\bar{D}$ for $c_0 \operatorname{Im} w > 0$.

Finally, for arbitrary $w \in \mathbf{C}$, and the linear fractional transformation Φ_w in (1.13), $\Phi_w(\bar{D}) \subseteq \bar{D}$ if and only if $e^{cw}H \subseteq H$, where H is the half-plane

$\mathrm{Re}[(1-\bar{\alpha}\beta)z] \leqslant 0$. This completes the proof of the theorem.

Let $\{\phi_t\}$ be an arbitrary one-parameter group of Möbius transformations of D. In order to expedite the use of (0.1) in the study of one-parameter groups of isometries of H^p, we note here that $\phi_2(\cdot\,,\cdot)$ has a continuous logarithm $L_\phi(\cdot\,,\cdot)$ on $\mathbf{R}\times\bar{D}$ such that $L_\phi(0,0)=0$ (the uniqueness of a logarithm meeting these requirements is immediate from connectedness). Indeed, if we denote $\phi_2(t,\cdot)$ (resp., $\phi_{22}(t,\cdot)$) by $\phi_t'(\cdot)$ (resp., $\phi_t''(\cdot)$), then straightforward reasoning proves the existence of $L_\phi(\cdot\,,\cdot)$ in the form

$$L_\phi(t,z)=\int_0^t \frac{d[\phi_u'(0)]}{du}\,\frac{1}{\phi_u'(0)}\,du+\int_0^z \frac{\phi_t''(\xi)}{\phi_t'(\xi)}\,d\xi \quad \text{for } t\in\mathbf{R},\ z\in\bar{D}. \tag{1.18}$$

From the connectedness of $\mathbf{R}\times\mathbf{R}\times\bar{D}$ we see that

$$L_\phi(t+s,z)=L_\phi(t,\phi_s(z))+L_\phi(s,z) \quad \text{for all } t\in\mathbf{R},\ s\in\mathbf{R},\ z\in\bar{D}. \tag{1.19}$$

If, further, $\{\phi_t\}$ is of type (i) or (ii), then, in the notation of Theorem (1.14), let H_Φ be the half-plane $\{w\in\mathbf{C}\colon \Phi_w(\bar{D})\subseteq\bar{D}\}$. In a fashion analogous to the foregoing we note that $\Phi_2(w,z)$ $(=\partial\Phi_w(z)/\partial z)$ has, on $H_\Phi\times\bar{D}$, a unique continuous logarithm vanishing at $(0,0)$. Denoting this logarithm by L_Φ, we have that

$$L_\Phi(w,z)=\int_0^w \frac{d[\Phi_\xi'(0)]}{d\xi}\,\frac{1}{\Phi_\xi'(0)}\,d\xi+\int_0^z \frac{\Phi_w''(\xi)}{\Phi_w'(\xi)}\,d\xi \quad \text{for } w\in H_\Phi,\ z\in\bar{D}. \tag{1.20}$$

The contour of the first integral on the right of (1.20) is in H_Φ. Moreover,

$$L_\Phi(w_1+w_2,z)=L_\Phi(w_1,\Phi_{w_2}(z))+L_\Phi(w_2,z) \quad \text{for all } w_1\in H_\Phi,\ w_2\in H_\Phi,\ z\in\bar{D}. \tag{1.21}$$

2. **One-parameter groups of isometries of H^p.** Let G be the set of all one-parameter groups of Möbius transformations of D. If $\{\phi_t\}\in G$ and $1\leqslant p<\infty$, then for $t\in\mathbf{R}$, $z\in\bar{D}$ we define $[\phi_t'(z)]^{1/p}$ to be $\exp[(1/p)L_\phi(t,z)]$. Also, let Γ_p be the set of all one-parameter groups of isometries of H^p which are continuous in the strong, but not in the uniform, operator topology. For each nonnegative integer n, let e_n be the function in H^∞ defined by $e_n(z)=z^n$.

(2.1) THEOREM. *If $1\leqslant p<\infty$, there is a one-to-one map M_p of $\mathbf{R}\times G$ into Γ_p. The M_p-image $\{T_t\}$ of $(\omega,\{\phi_t\})$ is given by*

$$(T_tf)(z)=e^{i\omega t}[\phi_t'(z)]^{1/p}f(\phi_t(z)) \quad \text{for } f\in H^p,\ z\in D,\ t\in\mathbf{R}. \tag{2.2}$$

If, further, $p \neq 2, M_p$ *maps* $\mathbf{R} \times G$ *onto* Γ_p.

PROOF. It is immediate from (0.1) and (1.19) that $\{T_t\}$ in (2.2) is a one-parameter group of isometries of H^p. Strong continuity of $\{T_t\}$ is easily verified by first noticing that if f is a polynomial, $T_t f$, as a function of t, is continuous from $\mathbf{R}$ to H^p. $\{T_t\}$ is not continuous in the uniform operator topology by [1, Theorem (2.4)(ii)]. The group $\{\phi_t\}$ can be recovered from $\{T_t\}$, since for each $t \in \mathbf{R}$, $\phi_t = (T_t e_1)/(T_t e_0)$ on D. It is now evident that M_p is one-to-one. On the other hand, if $p \neq 2$ and $\{U_t\} \in \Gamma_p$, then (see [1, Theorem (2.4)(i) and discussion leading up to it]) it follows from (0.1) that there are $\{\phi_t\} \in G$ and a function $\lambda(\cdot)$ from $\mathbf{R}$ into C such that

$$(2.3) \qquad (U_t f)(z) = \lambda(t)[\phi_t'(z)]^{1/p} f(\phi_t(z)) \quad \text{for } t \in \mathbf{R},\ f \in H^p,\ z \in D.$$

Taking f equal to e_0 in (2.3) shows that $\lambda(\cdot)$ is continuous on $\mathbf{R}$. For each $s \in \mathbf{R}$, $U_s = \lambda(s) T_s$, where $\{T_t\} \in \Gamma_p$ corresponds to $(0, \{\phi_t\})$ in (2.2). It follows that $\lambda(\cdot)$ is a continuous character of $\mathbf{R}$. This completes the proof.

REMARK. It is not difficult to show by example that the range of M_2 is a proper subset of Γ_2.

DEFINITION. If $\{T_t\} = M_p(\omega, \{\phi_t\})$, we shall call $\{\phi_t\}$ the conformal group associated with $\{T_t\}$, and ω the logarithmic index of $\{T_t\}$. We shall say that $\{T_t\}$ is of type (i), (ii), or (iii) according as $\{\phi_t\}$ is. Given any group $\{\phi_t\}$ of type (i) or (ii) with unique planar extension $\{\Phi_w\}$, we define $[\Phi_w'(z)]^{1/p}$ on $H_\Phi \times \bar{D}$ to be $\exp[(1/p)L_\Phi(w, z)]$.

(2.4) THEOREM. *If* $1 \leq p < \infty$, *and* $\{T_t\}$ *is a one-parameter group of* H^p*-isometries of type* (i) *or* (ii), *with associated conformal group* $\{\phi_t\}$, *then* $\{T_t\}$ *can be uniquely extended to a semigroup of bounded operators on* H^p, $\{T_w\}$, $w \in H_\Phi$, *such that* $\{T_w\}$ *is continuous on* H_Φ *in the strong operator topology and holomorphic on the interior of* H_Φ. *If* ω *is the logarithmic index of* $\{T_t\}$, *then* $\{T_w\}$ *is given by*:

$$(2.5) \qquad (T_w f)(z) = e^{i\omega w}[\Phi_w'(z)]^{1/p} f(\Phi_w(z)) \quad \textit{for } f \in H^p,\ z \in D,\ w \in H_\Phi.$$

Moreover, $\|T_w\| \leq e^{-\omega \operatorname{Im} w}$ *for all* $w \in H_\Phi$.

PROOF. Uniqueness follows from the observation that, by the Schwarz reflection principle, a continuous complex-valued function on H_Φ analytic on the interior of H_Φ and vanishing on the real axis must vanish identically. For each $w \in H_\Phi$ it follows by a result of Gabriel [5, p. 117] that the right-hand side of (2.5) with the factor $e^{i\omega w}$ deleted defines a linear transformation S_w of H^p into H^p such that $\|S_w\| \leq 1$. With the aid of (1.21) it is now clear that (2.5) defines a semigroup of bounded operators $\{T_w\}$, $w \in H_\Phi$, which extends $\{T_t\}$, $t \in \mathbf{R}$. To complete the proof it suffices to show that for each $f \in H^p$, and each continuous linear functional G on H^p, $G(S_w f)$ is, as a function of w, continuous on H_Φ and analytic on the interior of H_Φ. If $p > 1$, then the evaluation functionals at points of D span a norm-dense linear manifold in the dual space of H^p, and, since $\{S_w\}$ is uniformly bounded, it is enough to observe that if G is evaluation at a point of D, then $G(S_w f)$ has the desired properties. If $p = 1$, the proof reduces to the case where f is a polynomial, and is then easily completed by passage to the boundary.

DEFINITION. If $\{\phi_t\} \in G$ and $1 \leqslant p < \infty$, we shall denote $M_p(0, \{\phi_t\})$ by $\{T_t^{(p,\phi)}\}$, or, when there is no danger of confusion, by $\{T_t^{(\phi)}\}$. Further, we denote the invariance polynomial of $\{\phi_t\}$ by q_ϕ, and we define a closed linear operator $Q^{(p,\phi)}$, with both domain and range in H^p, as follows: its domain $\mathcal{D}(Q^{(p,\phi)})$ is $\{f \in H^p : q_\phi f' \in H^p\}$, and

$$(2.6) \qquad (Q^{(p,\phi)}f)(z) = q_\phi(z)f'(z) + \frac{1}{p}\, q'_\phi(z)f(z) \quad \text{for } f \in \mathcal{D}(Q^{(p,\phi)}),\ z \in D.$$

In a context where p is fixed, we shall write $Q^{(\phi)}$ instead of $Q^{(p,\phi)}$.

(2.7) THEOREM. *If* $\{\phi_t\} \in G$ *and* $1 \leqslant p < \infty$, *then the infinitesimal generator of* $\{T_t^{(p,\phi)}\}$ *is* $Q^{(p,\phi)}$.

PROOF. Let $A^{(\phi)}$ be the generator of $\{T_t^{(\phi)}\}$. If $f \in \mathcal{D}(A^{(\phi)})$ and $z \in D$, then

$$(A^{(\phi)}f)(z) = \frac{d}{dt}\Big|_{t=0} [(T_t^{(\phi)}f)(z)],$$

and it follows that $Q^{(\phi)}$ extends $A^{(\phi)}$. If $\{\phi_t\}$ is of type (i), then $Q^{(\phi)} = A^{(\phi)}$ by [1, Theorem (3.1)(i)]. If $\{\phi_t\}$ has the form (1.8) (resp., (1.9)), and if η is any complex number, then it is readily verified that the function F on D whose value at each z is given by

$$(z-\alpha)^{2(p^{-1}-1)} \exp\{\alpha\eta/ic(z-\alpha)\}$$

$$(\text{resp., by } (z-\alpha)^{1/p-\eta/c-1}(z-\beta)^{1/p+\eta/c-1})$$

has the property that for each analytic function f on D

$$(2.8) \qquad \frac{d(Fq_\phi f)}{dz} = -\,[\eta f - (q_\phi f' + q'_\phi f p^{-1})]F \quad \text{on } D.$$

In particular, if $\eta = c$, and f is in the null space of $cI - Q^{(\phi)}$, then there is a constant K such that $f = K/(Fq_\phi)$ on D. Since $1/(Fq_\phi)$ is not in H^p, we get that $cI - Q^{(\phi)}$ is one-to-one. Since the spectrum of the generator of a strongly continuous one-parameter group of Banach space isometries must be a subset of

$i\mathbf{R}$ [2, VIII.1.11], $cI - A^{(\phi)}$ maps $\mathcal{D}(A^{(\phi)})$ onto H^p. Because the one-to-one operator $cI - \mathcal{Q}^{(\phi)}$ extends the surjective operator $cI - A^{(\phi)}$, we conclude easily that $\mathcal{D}(\mathcal{Q}^{(\phi)}) = \mathcal{D}(A^{(\phi)})$.

3. **Spectral properties of** $\mathcal{Q}^{(p,\phi)}$. If $\{T_t\} = M_p(\omega, \{\phi_t\})$, then the infinitesimal generator of $\{T_t\}$ is $i\omega + \mathcal{Q}^{(p,\phi)}$. Thus, from the standpoint of spectral analysis, the study of such a generator reduces to the study of the corresponding operator $\mathcal{Q}^{(p,\phi)}$. We shall denote the spectrum of an operator A by $\Lambda(A)$.

(3.1) THEOREM. *Let* $\{\phi_t\} \in G$. *For* $1 \leqslant p < \infty$ *we have*:

(i) *If* $\{\phi_t\}$ *is of type* (i), *with the representation* (1.7), *then*: $\mathcal{Q}^{(p,\phi)}$ *has compact resolvent function*; $\Lambda(\mathcal{Q}^{(p,\phi)}) = \{ic(n + p^{-1}): n = 0, 1, 2, \cdots\}$; *for* $n = 0, 1, 2, \cdots, ic(n + p^{-1})$ *is an eigenvalue of* $\mathcal{Q}^{(p,\phi)}$, *and the eigenmanifold corresponding to* $ic(n + p^{-1})$ *is one-dimensional.*

(ii) *If* $\{\phi_t\}$ *is of type* (ii), *with the representation* (1.8), *then* $\Lambda(\mathcal{Q}^{(p,\theta)}) = \{-i\lambda: \lambda c \geqslant 0\}$, *and* $\mathcal{Q}^{(p,\phi)}$ *has no eigenvalues.*

(iii) *If* $\{\phi_t\}$ *is of type* (iii), *then* $\Lambda(\mathcal{Q}^{(p,\phi)}) = i\mathbf{R}$, *and* $\mathcal{Q}^{(p,\phi)}$ *has no eigenvalues.*

PROOF. In all cases we have $\Lambda(\mathcal{Q}^{(\phi)}) \subseteq i\mathbf{R}$ by virtue of Theorem (2.7). Assertion (i) is contained in [1, Theorem (3.1)], and is listed above for comparison with (ii) and (iii). If $\{\phi_t\}$ is of type (ii) or (iii), and $\lambda \in \mathbf{R}$, then, just as in the proof of Theorem (2.7), an argument based on (2.8) (with $\eta = i\lambda$ in this instance) shows that $i\lambda - \mathcal{Q}^{(\phi)}$ is one-to-one.

Next, let $\{\phi_t\}$ be of type (iii), with the representation (1.9). Let $\lambda \in \mathbf{R}$, and for each $w \in \mathbf{C}$ with $\mathrm{Re}(w) > -1/p$, let f_w be defined on D by $f_w(z) = (z - \alpha)^w$. Then $f'_w(z) = w(z - \alpha)^{-1} f_w(z)$, and we have that $f_w \in \mathcal{D}(\mathcal{Q}^{(\phi)})$.

$$
\begin{aligned}
&[(i\lambda - \mathcal{Q}^{(\phi)})f_w](z) \\
&\quad = f_w(z)\left[i\lambda - \frac{wc}{\alpha - \beta}(z - \beta) - \frac{c}{p(\alpha - \beta)}(z - \alpha) - \frac{c}{p(\alpha - \beta)}(z - \beta)\right] \\
&\quad = \frac{c}{\beta - \alpha}\left(w + \frac{2}{p}\right)(z - \alpha)f_w(z) - c(w - w_0)f_w(z), \quad \text{where } w_0 = -\frac{1}{p} + \frac{i\lambda}{c}.
\end{aligned}
$$

As $w \to w_0$ (with $\mathrm{Re}(w) > -1/p$), $\|(z - \alpha)f_w\|_\infty = O(1)$, while $\|f_w\|_p \to +\infty$. Thus $(\|(i\lambda - \mathcal{Q}^{(\phi)})f_w\|_p / \|f_w\|_p) \to 0$, and the proof of (iii) is complete.

If $\{\phi_t\}$ has the form (1.8), let $\{T_w\}$, $w \in H_\Phi$, be the extension of $\{T_t^{(\phi)}\}$, $t \in \mathbf{R}$, furnished by Theorem (2.4). In this instance H_Φ is $\{w \in \mathbf{C}: c \operatorname{Im} w \leqslant 0\}$, and (since $\omega = 0$) each T_w is a contraction operator. Let R be $\{w \in \mathbf{C}: \mathrm{Re}(w) \geqslant 0\}$. If $c > 0$ (resp., $c < 0$) define $\{U_w\}$, $w \in R$, by setting $U_w = T_{-iw}$ (resp., $U_w = T_{iw}$). By [6, Theorem 17.9.2] the infinitesimal generator A of $\{U_t\}$, $t \geq 0$, is $(-i)\mathcal{Q}^{(\phi)}$ (resp., $i\mathcal{Q}^{(\phi)}$). Since, by [2, VIII.1.11], $\Lambda(A) \subseteq \{\mu \in \mathbf{C}: \operatorname{Re}(\mu) \leq 0\}$, we get that $\Lambda(\mathcal{Q}^{(\phi)}) \subseteq \{-i\lambda: c\lambda \geq 0\}$. Conversely, if $c\lambda \geq 0$, for $t > -1/p$ define $f_t \in H^p$ by setting $f_t(z) = (z-\alpha)^t \exp(\lambda\alpha/c(z-\alpha))$. Then $f_t \in \mathcal{D}(\mathcal{Q}^{(\phi)})$, and

$$[(i\lambda + \mathcal{Q}^{(\phi)})f_t](z) = ic\bar{\alpha}\left(t + \frac{2}{p}\right)(z-\alpha)f_t(z).$$

As $t \longrightarrow -1/p$ from the right, clearly $\|(z-\alpha)f_t\|_\infty = O(1)$ and $\|f_t\|_p \longrightarrow +\infty$. This completes the proof of the theorem.

REMARK. If $\{\phi_t\}$ is of type (ii) or (iii), then $\mathcal{Q}^{(\phi)}$ does not have compact resolvent function, since $\Lambda(\mathcal{Q}^{(\phi)})$ is uncountable.

DEFINITION. For $1 \leq p < \infty$ and $\{\phi_t\} \in G$, let $\Delta^{(p,\phi)}$ be $\{\mu \in \Lambda(\mathcal{Q}^{(p,\phi)})$: the range of $\mu - \mathcal{Q}^{(p,\phi)}$ is dense in $H^p\}$.

(3.2) THEOREM. (i) *If* $\{\phi_t\}$ *is of type* (i), *then* $\Delta^{(p,\phi)}$ *is empty for* $1 \leq p < \infty$.

(ii) *If* $\{\phi_t\}$ *is of type* (ii), *then* $\Delta^{(p,\phi)} = \Lambda(\mathcal{Q}^{(p,\phi)})$ *for* $1 < p < \infty$, *and* $\Delta^{(1,\phi)} = \{0\}$.

(iii) *If* $\{\phi_t\}$ *is of type* (iii), *then* $\Delta^{(p,\phi)} = \Lambda(\mathcal{Q}^{(p,\phi)})$ *for* $1 < p < \infty$, *and* $\Delta^{(1,\phi)}$ *is empty.*

PROOF. For $1 \leq p < \infty$, $\{\phi_t\} \in G$, and $\lambda \in \mathbf{R}$, it is clear that $i\lambda - \mathcal{Q}^{(p,\phi)}$ generates a strongly continuous one-parameter group of isometries of H^p. The desired conclusions for $p > 1$ are now immediate from Theorem (3.1) and the following general fact (see [2, VIII.7.2]): If A generates a uniformly bounded strongly continuous semigroup of operators on a reflexive Banach space X, then X is the direct sum of the null space of A and the closure of the range of A.

To complete the proof of (i), let $\{\phi_t\}$ have the form (1.7), let n be a nonnegative integer, and suppose $[ic(n+1) - \mathcal{Q}^{(1,\phi)}]f = g$. It is easy to verify that on the region obtained by deleting τ from D

$$(3.3)\qquad \begin{aligned} -\frac{(z-\bar{\tau}^{-1})^{n+2}g(z)}{(z-\tau)^n q_\phi(z)} &= \frac{d}{dz}\left[\frac{(z-\bar{\tau}^{-1})^{n+2}}{(z-\tau)^n} f(z)\right], && \text{if } \tau \neq 0; \\ -\frac{g(z)}{z^n q_\phi(z)} &= \frac{d}{dz}\left(\frac{f(z)}{z^n}\right), && \text{if } \tau = 0. \end{aligned}$$

The right-hand side of each equation in (3.3) has a vanishing contour integral around any circle $|z| = r$, $|\tau| < r < 1$. Thus we get for $\tau \neq 0$ (resp., $\tau = 0$) $((z-\bar{\tau}^{-1})^{n+1}g(z))^{(n)}(\tau) = 0$ (resp., $g^{(n)}(0) = 0$). Since the range of $ic(n+1) - \mathcal{Q}^{(1,\phi)}$ is contained in the null space of a nonzero continuous linear functional on H^1, the desired conclusion follows.

Next let $\{\phi_t\}$ have the form (1.8). If $\lambda c > 0$ and $g = -(i\lambda + \mathcal{Q}^{(1,\phi)})f$, then by (2.8) there is a constant K such that for all $z \in D$

$$f(z) = -\frac{1}{F(z)q_\phi(z)}\left[\int_0^z g(\xi)F(\xi)\,d\xi + K\right], \tag{3.4}$$

where $F(z) = \exp(-\lambda\alpha/c(z-\alpha))$ (we shall regard F as being defined on $\bar{D}\backslash\{\alpha\}$ by this formula). We observe that $|F(z)| = \exp(\lambda/2c)$ for $z \in C$, $z \neq \alpha$. However, F does not belong to $H^\infty(D)$, whereas the reciprocal of F does. As z approaches radially any point z_0 of $C\backslash\{\alpha\}$, the integral in (3.4) tends to a limit (denoted $\int_0^{z_0} g(\xi)F(\xi)\,d\xi$). Since $g(e^{i\theta})F(e^{i\theta})$ is in L^1 of C, it is easy to see that $\int_0^z g(\xi)F(\xi)\,d\xi$ has one-sided limits as $z \to \alpha$ on C. Because $f(e^{i\theta})$ is in L^1 of C, both one-sided limits must be $-K$. It follows that g is in the null space of the continuous linear functional $y(\cdot)$ on H^1 defined by $y(h) = \int_{|\xi|=1} h(\xi)F(\xi)\,d\xi$. Note that since $1/F$ does not have an inverse in the algebra $H^\infty(D)$, the restriction of F to C, which is the reciprocal of the boundary function of $1/F$, cannot be in $H^\infty(C)$. However, if y were identically zero, this restriction of F would be in $H^\infty(C)$. It follows that $-i\lambda \notin \Delta^{(1,\phi)}$ for $c\lambda > 0$. On the other hand, if h is analytic on D with $h(z) = O(|z-\alpha|^2)$, then, as a consequence of Hardy's inequality, h has a primitive G on D such that $G(z) \to 0$ as $z \to \alpha$. It is easy to see that $G(z)/q_\phi(z) = O(|z-\alpha|)$ and $\mathcal{Q}^{(1,\phi)}(G/q_\phi) = h$. Since $(z-\alpha)$ is outer, $O \in \Delta^{(1,\phi)}$.

Suppose next that $\{\phi_t\}$ has the form (1.9), that $\lambda \in \mathbf{R}$, and that $(ic\lambda - \mathcal{Q}^{(1,\phi)})f = g$. As before, equation (3.4) holds on D, and F now has the form $(z-\alpha)^{-i\lambda}(z-\beta)^{i\lambda}$. In this instance both F and its reciprocal are in H^∞, and, by virtue of Hardy's inequality, the integral appearing in (3.4) has, as a function of z, a continuous extension to $\bar{D}$. It is easy to see that we must have

$$\int_0^\alpha g(\xi)F(\xi)\,d\xi = \int_0^\beta g(\xi)F(\xi)\,d\xi = -K.$$

Thus $\int_\alpha^\beta g(\xi)F(\xi)\,d\xi = 0$ (either arc of C from α to β may be used for this path integral by [3, Theorem (3.6)]). (It is easy to see that the range of $(ic\lambda - \mathcal{Q}^{(1,\phi)})$ contains $\{G \in \bigcup_{1<p\leq\infty} H^p : \int_\alpha^\beta G(\xi)F(\xi)\,d\xi = 0\}$; however, we do not need this fact, and omit its proof.) Let us define a continuous linear functional $y(\cdot)$ on H^1 by the formula $y(h) = \int_\alpha^\beta h(\xi)F(\xi)\,d\xi$. Note that $y(1/F) = \beta - \alpha \neq 0$. This completes the proof of the theorem.

4. **Concluding remarks concerning ranges.** In this section we shall examine the range of $(\mu - \mathcal{Q}^{(p,\phi)})$, where $\mu \in \Delta^{(p,\phi)}$. Our aim will be to find a (more-or-less concretely described) dense subset of H^p which is also a subset of the range of $(\mu - \mathcal{Q}^{(p,\phi)})$. For $p = 1$ this setting occurs if and only if $\{\phi_t\}$ is of type (ii) and $\mu = 0$; for this case a specific dense subset of H^1 which is also a subset

of the range of $Q^{(1,\phi)}$ was found in the proof of Theorem (3.2).

(4.1) THEOREM. *Suppose that* $1 < p < \infty$, *and* $\{\phi_t\}$ *is of type* (iii) *with the representation* (1.9). *For each* $\lambda \in \mathbf{R}$,

$$\left\{g \in \bigcup_{p<s\leqslant\infty} H^s\colon \int_\alpha^\beta g(\xi)(\xi-\alpha)^{-i\lambda}(\xi-\beta)^{i\lambda}[(\xi-\alpha)(\xi-\beta)]^{(1/p)-1}\,d\xi = 0\right\}$$

is dense in H^p, *and is a subset of the range of* $(ic\lambda - Q^{(p,\phi)})$.

PROOF. We note that in the above contour integral, we are multiplying g by the function F appearing in (2.8) (for $\eta = ic\lambda$). Let $g \in H^{p_1}$ for some $p_1 > p$, and observe that $Fg \in H^{1+\delta}$ for some $\delta > 0$. If $\int_\alpha^\beta g(\xi)F(\xi)\,d\xi = 0$, then the function of $z \in D\ \int_\alpha^z g(\xi)F(\xi)\,d\xi$ $(= \int_\beta^z g(\xi)F(\xi)\,d\xi)$ is $O(|z-\alpha|^{\delta/(1+\delta)})$ and $O(|z-\beta|^{\delta/(1+\delta)})$. From this we see that $-(F(z)q_\phi(z))^{-1}\int_\alpha^z g(\xi)F(\xi)\,d\xi$ is in H^p. By (2.8) $(ic\lambda - Q^{(p,\phi)})$ sends the latter function into g.

To complete the proof let X be the normed linear space $\bigcup_{p<s\leqslant\infty} H^s$ (with H^p-norm). Define the linear functional G on X by $G(g) = \int_\alpha^\beta g(\xi)F(\xi)\,d\xi$. We show that the kernel of G is dense in X by proving that G is not continuous. For $t > -1/p$, define $f_t \in X$ by

$$f_t(z) = (z-\alpha)^{i\lambda}(z-\beta)^{-i\lambda}(z-\alpha)^t(z-\beta)^{1-1/p}.$$

Then

$$G(f_t) = \int_\alpha^\beta (\xi-\alpha)^{t+p^{-1}-1}\,d\xi = (\beta-\alpha)^{t+p^{-1}}(t+p^{-1})^{-1}.$$

But as $t \longrightarrow -1/p$, $\|f_t\|_p = O((t+p^{-1})^{-1/p})$.

The following lemma will be useful in studying the type (ii) case.

(4.2) LEMMA. *If* $f \in H^\infty$, $\alpha \in C$, *and* $\lambda \leqslant 0$, *then*

$$\left|\int_{-\alpha}^z e^{\lambda\alpha/(\xi-\alpha)}f(\xi)\,d\xi\right| = O(|e^{\lambda\alpha/(z-\alpha)}|), \quad \textit{for} \quad |z| < 1.$$

PROOF. For convenience we shall, without loss of generality, take α equal to 1. Integrate from -1 along $[-1, 1]$ to the point ξ_0 on the circle $\operatorname{Re} 1/(\xi-1) = \operatorname{Re} 1/(z-1)$, and then along the minor arc of this circle from ξ_0 to z. Since $\lambda/(x-1)$ increases on $[-1, 1)$, the absolute value of the first integral does not exceed $2\|f\|_\infty|e^{\lambda/(z-1)}|$. The absolute value of the second integral does not exceed $\pi\|f\|_\infty|e^{\lambda/(z-1)}|$.

(4.3) THEOREM. *Suppose that* $1 < p < \infty$, *and* $\{\phi_t\}$ *is of type* (ii) *with the representation* (1.8). *For each* $\lambda \leqslant 0$,

$$\left\{g \in H^\infty : g(z) = O(|z-\alpha|^2) \text{ and } \int_{|\xi|=1} g(\xi)(\xi-\alpha)^{2(p^{-1}-1)} e^{(\lambda\alpha)/(\xi-\alpha)}\, d\xi = 0\right\}$$

is dense in H^p, *and is a subset of the range of* $(ic\lambda - Q^{(p,\phi)})$.

PROOF. Let N be the set in the statement of the theorem, and observe that in the (absolutely convergent) contour integral used to define N, g is multiplied by the function F appearing in (2.8) (for $\eta = ic\lambda$). For any $g \in N$, $g(z)(z-\alpha)^{2((1/p)-1)}$ is in $H^\infty(D)$. Define G on D by

$$G(z) = \int_{-\alpha}^{z} g(\xi)F(\xi)\, d\xi + \int_{\alpha}^{-\alpha} g(\xi)F(\xi)\, d\xi$$

(by assumption on g, the latter integral is the same along either semicircle from α to $-\alpha$). By Lemma (4.2),

$$\frac{G(z)}{F(z)q_\phi(z)} = O(|z-\alpha|^{-2/p}) \quad \text{for } |z| < 1.$$

Clearly, then, G/Fq_ϕ is in $H^{1/3}$. The boundary function of G/Fq_ϕ is

$$\frac{1}{Fq_\phi}\left(\int_{\alpha}^{z} g(\xi)F(\xi)\, d\xi\right),$$

where the integral is taken along either arc of C from α to z. It is easy to see that this boundary function is $O(|z-\alpha|)$ for $|z| = 1$, and it follows by [3, Theorem (2.7)] that G/Fq_ϕ is in $H^\infty(D)$. By (2.8), $(ic\lambda - Q^{(p,\phi)})(-G/Fq_\phi) = g$.

Let X be the set of all analytic functions on D which are $O(|z-\alpha|^2)$. If $\lambda = 0$, then by the Cauchy integral theorem $N = X$, and the density of N is obvious in this case. If $\lambda < 0$, we regard X as a normed linear space with the H^p-norm, and we complete the proof of the theorem by showing that the linear functional $f \mapsto \int_{|\xi|=1} f(\xi)F(\xi)\, d\xi$ is not continuous on X. Without loss of generality we take $\alpha = 1$. If the functional were continuous, there would be a function W in $L^{p'}(d\theta)$ (where p' is conjugate to p) such that

$$\int_{|\xi|=1} g(\xi)(\xi-1)^2F(\xi)\, d\xi = \int_{|\xi|=1} g(\xi)(\xi-1)^2 W(\xi)\, d\xi \quad \text{for all } g \in H^\infty.$$

It follows, since $(\xi-1)^2F(\xi)$ and $(\xi-1)^2W(\xi)$ are integrable, that there is a function $h \in H^1$ such that $(\xi-1)^2F(\xi)\, d\xi = (\xi-1)^2W(\xi)\, d\xi + h(\xi)\, d\xi$. This gives

$$(\xi-1)^{2/p}\, d\xi = e^{-\lambda/(\xi-1)}(\xi-1)^2W(\xi)\, d\xi + e^{-\lambda/(\xi-1)}h(\xi)\, d\xi. \tag{4.4}$$

Multiply (4.4) by $(2\pi i)^{-1}(\xi - r)^{-3}$, $0 < r < 1$, and then integrate both sides around C. The left side is $p^{-1}((2/p) - 1)(r - 1)^{2((1/p)-1)}$. For any $b > 0$, the second term on the right is $O((1 - r)^b)$, while standard estimates show that the first term on the right is $O((1 - r)^{-1+1/p})$. By letting $r \longrightarrow 1$ we obtain the desired contradiction if $p \neq 2$. If $p = 2$, multiply (4.4) by $(2\pi i)^{-1}(\xi - r)^{-3}(\xi - 1)^{1/4}$. As before, we obtain a contradiction.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801